

On active drains and causality

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The concept of an *active drain* has been used recently to provide an optical element which manages to perfectly sink incoming electromagnetic radiation. Here I show that without prior knowledge of the incoming signal, an element attempting to respond as an active localized drain cannot succeed.

There is a current interest in using time-reversed sources as a kind of perfect sink (or “drain”) designed to exactly cancel incoming signals [1–3]. These are active, not passive devices, and need to be perfectly matched to the incoming radiation – i.e. both spatially *and* temporally matched. Here I leave the spatial properties to one side and instead focus solely on the temporal behaviour. I ask the question: *If the active drain does not know what signal is about to arrive, can it exactly cancel the incoming field regardless?*

Thus here I consider a more general case than was considered in the three cases mentioned above (i.e. [1–3]), where knowledge of the incoming radiation is known in advance and so built into the device. This can be seen from how De Rosny et al.’s acoustic sink is explicitly constructed from a time-reversal of the known source, and how Chong et al.’s time-reversed laser [2] demands “specific conditions of coherent monochromatic illumination”. For the mirrored version of Maxwell’s fisheye lens [1], whilst its claimed perfect imaging capabilities are a matter of ongoing debate [4–8], nevertheless all parties seem to agree that *if* it can do so[17], it relies on the presence of an active drain.

I. THE RESPONSE OF AN ACTIVE DRAIN

In a more general environment where the properties of the incoming signal properties are not predetermined and already designed into the active drain, the drain element will have to follow some causal response driven by the source $S(t)$. In the case of electromagnetism, the drain response might in essence be just some more complicated version of an ordinary dielectric polarization, which might e.g. follow a Lorentz or Drude form [9]. I therefore start by writing down a general differential equation for the response of the drain element $P(t)$ using a summed series of time derivatives (where $\partial_t \equiv d/dt$). This then allows as wide as possible a range of responses, and is unlikely to exclude behaviours that might turn out to be useful[18]. This general response is

$$\sum_{n=0}^N T_n \partial_t^n P(t) = \sum_{m=0}^{N-1} a_m \partial_t^m S(t - \tau). \quad (1)$$

Here the T_n control the dynamics of the drain element, while the a_m allow for the coupling between the drain element and the properties of the incident signal field S [19]. The drain element experiences a signal field delayed by τ from that produced by the source according to the path length[20]. In the description here I do not consider how propagation might affect the source signal before it reaches the location of the drain. Here, S defines what that signal has become when it reaches the drain, although I leave in the time-delay τ to emphasize the retardation between the generation of the signal and its reception. Note that we cannot have an $n = 0$ term by itself, because contributions like “ $T_0 P = a_0 S$ ” specify an identity, not a causal relationship; more generally the derivatives on the LHS should be at least one order higher than those on the right. In the frequency domain, where time derivatives convert to factors of $-i\omega$, eqn. (1) becomes

$$\sum_{n=0}^N (-i\omega)^n T_n P(\omega) = \sum_{m=0}^{N-1} a_m S(\omega) e^{i\omega\tau}. \quad (2)$$

This can then be rearranged into $P(\omega) = f(\omega) S(\omega) e^{i\omega\tau}$ for a response function f which is a rational function of frequency ω , i.e.

$$f(\omega) = \frac{\sum_{m=0}^{N-1} (-i\omega)^m a_m}{\sum_{n=0}^N (-i\omega)^n T_n}. \quad (3)$$

For a finite maximum N , appropriate choices of T_n and a_m should ensure that f has sufficiently simple poles below some line parallel to the real ω axis [10]. Further, a non-zero T_N and the restriction of the m summation to a maximum of $N - 1$ guarantees that f has the appropriate limiting behaviour of at least $\sim \omega^{-1}$ as $\omega \rightarrow \infty$. Thus P can be shown to remain causal in the sense that it satisfies the Kramers Kronig relations [11].

However, in addition to P ’s response to the driving from S , it also needs to behave like a drain: i.e. we need that $P(t) = -S(t - \tau)$ or $P(\omega) = -S(\omega) e^{i\omega\tau}$. Without this perfect correspondence, the active drain will not sink the incoming signals from the source. Note that the cancellation of fields by a drain is not “loss” in the sense of some irreversible dissipation, but is instead a carefully arranged *destructive interference*. Hence, for the drain to work, we need

$$\sum_{n=0}^N (-i\omega)^n T_n = - \sum_{m=0}^{N-1} (-i\omega)^m a_m, \quad (4)$$

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which amounts to demanding that two polynomials of *different order* are somehow equal. Crucially, since the T_n and a_m are fixed parameters of the response function, the equality will not hold over some finite frequency interval, but only at specific intersection frequencies.

Now consider an electromagnetic drain whose behaviour is expressed in terms of dielectric polarization P , with a signal arriving through a dispersionless linear medium as an electric field E , and impinging from all directions on the drain (such as the image point of a Maxwell's fisheye lens [12] as discussed by Leonhardt [1]). Here we need the drain polarization $P(t) = -\epsilon E(t)$, so that the displacement field $D = \epsilon E + P$ vanishes. This follows because for situations without magnetic sources (as considered here), the second order wave equation can be written as $\nabla^2 E = -\mu \partial_t^2 D$. Thus if $D = 0$ is always true at the drain position, then by simple integration, E at the drain position can at most have a linear spatial variation offset by some constant background. For a symmetric situation both linear and background terms vanish – the linear is incompatible with symmetry, and the offset must remain at the value established before the signal field arrived at the drain (i.e. zero). In simple terms, without the drain's polarization P , an incoming signal would pass through the drain point and become an outgoing wave; but the drain polarization creates a wave anti-phase to the outgoing signal, causing perfect cancellation: only the incoming signal field survives. In asymmetric cases, such as where a field impinges onto (e.g.) the left hand side of a drain designed to cancel only the outward-going fields on the right, $D = -\epsilon E$ is then the appropriate criterion[21]. This can be shown using e.g. directional decompositions of the wave equation [13, 14].

Note that here I have chosen a relatively easy task, since I consider drains that only match the temporal properties of a signal at a single point. In contrast, drains that aim to e.g. cancel a signal over some spatial region, or minimise reflections and transmission, would suffer more constraints and so be either harder to implement or suffer worse performance.

II. AN EXAMPLE

Consider a device designed to act like a point-like active drain with a frequency independent response near its operating frequency ω_x . This behaviour can be satisfied perhaps most simply by requiring a response function quartic in time derivatives. To make the example more concrete, assume this is an electromagnetic problem in a medium with a linear background permittivity of ϵ . Therefore for incident signal (electric) field E and local response (dielectric polarization) field P , we have $T_4 \partial_t^4 P + T_2 \partial_t^2 P + \gamma \partial_t P = \epsilon E$, where I have used γ instead of T_1 to indicate the loss time-scale of the device

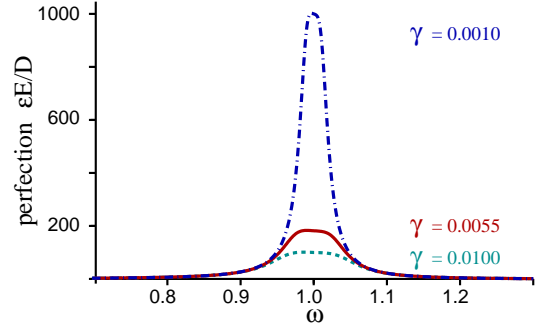


FIG. 1: The spectral “perfection” $|\epsilon E/D|$ of the active drain model as calculated from eqn. (6), for $T_4 = 1$, $T_2 = 2$, and three values of loss γ . A value over 100 implies signal cancellation to within 1%.

response. In the frequency domain, the response is

$$[T_4 \omega^4 - T_2 \omega^2 - \gamma \omega] P = \epsilon E. \quad (5)$$

A “perfect drain” condition occurs when these two fields (E and P) exactly cancel (i.e. $P = -\epsilon E$), giving a null displacement field $D = \epsilon E + P = 0$. Thus

$$\frac{D}{\epsilon E} = \frac{\epsilon E + P}{\epsilon E} = \frac{(T_4 \omega^4 - T_2 \omega^2 - \gamma \omega) + 1}{(T_4 \omega^4 - T_2 \omega^2 - \gamma \omega)}. \quad (6)$$

This means that if $T_4 \omega^4 - T_2 \omega^2 - \gamma \omega + 1 \simeq 0$, we will be close to having constructed an active drain. Further, if the operating frequency ω_x is chosen to be an extremal value of $T_4 \omega^4 - T_2 \omega^2$, then the frequency response will be nearly flat; such an extrema occurs at $\omega^2 = T_2/2T_4$. This has $T_4 \omega^4 - T_2 \omega^2 = T_2^2/4T_4$; so to get the desired behaviour at ω_x we need $T_2^2 = 4T_4$. The performance of the drain near ω_x will be good if

$$|T_4 \omega^4 - T_2 \omega^2 - \gamma \omega| \ll 1. \quad (7)$$

In the vicinity of ω_x , i.e. at $\omega = \omega_x + \delta$, we have that the LHS of the inequality in eqn. (7) varies as

$$|T_4 \omega^4 - T_2 \omega^2 - \gamma \omega| \simeq |2T_2 \delta^2 - \gamma \omega_x| = \sqrt{4T_2 \delta^4 + \gamma^2 \omega_x^2}, \quad (8)$$

so that we want both $(\delta/\omega_x)^2 \ll 1/2T_2\omega_x^2$ and $\gamma\omega_x \ll 1$. Thus the device will only work for low loss and over a bandwidth much smaller than its designed operating frequency; sample results are shown on fig. 1. This situation is similar to that regarding causal constraints on negative refractive index or perfect lenses [15, 16].

III. CONCLUSION

Causality restricts the perfect operation of an active drain to arriving signals that happen to match a *pre-specified* frequency of operation. However, a further complication is that no realistic source is exactly single frequency, it is only ever approximately so. This is

because even if we could eliminate unwanted frequency fluctuations in the source, the process of switching it on (or off) necessarily involves additional frequency components. Thus, since no more than a single frequency component of the signal could ever be canceled out, an active drain will *never* be perfect – except, of course, if causality is side-stepped by agreeing in advance on the signal and its timings (as in [1–3]).

The example given shows that an approximate ac-

tive drain can be achieved over a finite (albeit small) bandwidth, which in well controlled experimental situations should suffice to demonstrate some basic principles. However, outside carefully pre-arranged or restricted circumstances, an active drain cannot be guaranteed to work.

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 - [16] J. Skaar (2010), URSI EMTS conference, Berlin 2010, arXiv:1008.2874.
 - [17] And whether or not such a process still constitutes what might normally be described as “imaging”.
 - [18] As pointed out by the second reviewer, in addition to allowing derivatives of the source term, it is also legitimate to consider further time-delayed properties of the source.
 - [19] E.g. for a Lorentz response, we would have $T_2 = 1$, $T_1 = \gamma$ (loss), $T_0 = \omega_0^2$, and only $a_0 \neq 0$; the signal $S(t)$ would be replaced by the driving from the local value of the electric field.
 - [20] E.g. in the mirrored fisheye lens of [1], all optical path lengths from source to drain are the same, and correspond to a phase shift of $\phi = \omega\tau = 20\frac{1}{4}\pi$.
 - [21] This is because the drain polarization must radiate in both directions, but only the right hand part cancels the outward signal field.